LA NOTAZIONE "↑" DI KNUTH

$$a \times b = \underbrace{a + a + \cdots + a}_{b \text{ copies of } a}$$

Es. $$4 \times 3 = \underbrace{4 + 4 + 4}_{3 \text{ copies of } 4} = 12$$

$$a \uparrow b = a^b = \underbrace{a \times a \times \cdots \times a}_{b \text{ copies of } a}$$

Es. $$4 \uparrow 3 = 4^3 = \underbrace{4 \times 4 \times 4}_{3 \text{ copies of } 4} = 64$$

$$a \uparrow\uparrow b = {}^{b}a = \underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{b \text{ copies of } a} = \underbrace{a \uparrow (a \uparrow (\ldots \uparrow a))}_{b \text{ copies of } a}$$

(ASSOCIATIVITÀ A DESTRA)

ESEMPI

$$4 \uparrow\uparrow 3 = {}^{3}4 = \underbrace{4^{4^{4}}}_{3 \text{ copies of } 4} = \underbrace{4 \uparrow (4 \uparrow 4)}_{3 \text{ copies of } 4} = 4^{256} \approx 1.3 \times 10^{154}$$

$$3 \uparrow\uparrow 2 = 3^{3} = 27$$

$$3 \uparrow\uparrow 3 = 3^{3^{3}} = 3^{27} = 7{,}625{,}597{,}484{,}987$$

$$3 \uparrow\uparrow 4 = 3^{3^{3^{3}}} = 3^{7625597484987}$$

$$3 \uparrow\uparrow 5 = 3^{3^{3^{3^{3}}}} = 3^{3^{7625597484987}}$$

$$a \uparrow\uparrow\uparrow b = \underbrace{a \uparrow\uparrow (a \uparrow\uparrow (\ldots \uparrow\uparrow a))}_{b \text{ copies of } a}$$

(ASSOCIATIVITA' A DESTRA)

$$3 \uparrow\uparrow\uparrow 2 = 3 \uparrow\uparrow 3 = 3^{3^3} = 3^{27} = 7,625,597,484,987$$

$$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3) = \underbrace{3 \uparrow 3 \uparrow \ldots \uparrow 3}_{3 \uparrow 3 \uparrow 3 \text{ copies of } 3} = \underbrace{3 \uparrow 3 \uparrow \ldots \uparrow 3}_{7,625,597,484,987 \text{ copies of } 3}$$

$$a \uparrow\uparrow\uparrow\uparrow b = \underbrace{a \uparrow\uparrow\uparrow (a \uparrow\uparrow\uparrow (\ldots \uparrow\uparrow\uparrow a))}_{b \text{ copies of } a}$$

(ASSOCIATIVITA' A DESTRA)

$$a \underbrace{\uparrow\uparrow\dots\uparrow}_{n} b = \underbrace{a \underbrace{\uparrow\dots\uparrow}_{n-1} a \underbrace{\uparrow\dots\uparrow}_{n-1} a \dots a \underbrace{\uparrow\dots\uparrow}_{n-1} a}_{b \text{ copies of } a}$$

(ASSOCIATIVITA' A DESTRA)

SI PONE

$$\underbrace{\uparrow\uparrow\dots\uparrow}_{n} \equiv \uparrow^{(n)}$$

(ASSOCIATIVITA' A DESTRA)

QUINDI:

$$a \uparrow^{(n)} b = \underbrace{\left(a \uparrow^{(n-1)} \left(a \uparrow^{(n-1)} \left(a \dots \left(a \uparrow^{(n-1)} a\right) \dots\right)\right)\right)}$$

FUNZIONE DI ACKERMANN

• SI CONSIDERI LA SEGUENTE DEFINIZIONE PER RICORSIONE:

$$\begin{cases} \psi(0,y) = \begin{cases} 1 & \text{SE} \quad y=0 \\ 2 & \text{SE} \quad y=1 \\ y+2 & \text{SE} \quad y>1 \end{cases} \\ \psi(x,0) = 1 \\ \psi(x+1,y+1) = \psi(x,\psi(x+1,y)) \end{cases}$$

$$\left[\psi_{x+1}(y+1) = \psi_x(\psi_{x+1}(y))\right]$$

• CON LA NOTAZIONE $\psi_x(y) \equiv \psi(x,y)$ ABBIAMO:

$$\psi_{x+1}(y+1) = \psi_x(\psi_{x+1}(y)) = \psi_x(\psi_x(\psi_{x+1}(y-1))) = \psi_x^{(2)}(\psi_{x+1}(y-1))$$
$$= \psi_x^{(3)}(\psi_{x+1}(y-2)) = \dots = \psi_x^{(y+1)}(\psi_{x+1}(0)) = \psi_x^{(y+1)}(1).$$

• SE PONIAMO $\psi^{(0)}_{x+1}(y) = y$, PER OGNI $x, y \in \mathbb{N}$,

SI HA: $\boxed{\psi_{x+1}(y) = \psi_x^{(y)}(1)}$

SI HA:

$$\boxed{\psi_0(y) = \begin{cases} 1 & \text{se } y = 0 \\ 2 & \text{se } y = 1 \\ y+2 & \text{se } y > 1 \end{cases}}$$

SE $y > 0$,

$$\psi_0^{(y)}(1) = \psi_0^{(y-1)}(\psi_0(1)) = \psi_0^{(y-1)}(2)$$

$$= \psi_0^{(y-2)}(2+2) = \psi_0^{(y-2)}(2\cdot 2)$$

$$= \psi_0^{(y-3)}(2\cdot 2+2) = \psi_0^{(y-3)}(2\cdot 3)$$

$$= \dots = \psi_0^{(0)}(2\cdot y) = 2y$$

QUINDI

$$\boxed{\psi_1(y) = \psi_0^{(y)}(1) = \begin{cases} 1 & \text{se } y = 0 \\ 2y & \text{se } y > 0 \end{cases}}$$

SE $y > 0$,

$$\psi_1^{(y)}(1) = \psi_1^{(y-1)}(\psi_1(1)) = \psi_1^{(y-1)}(2)$$
$$= \psi_1^{(y-2)}(\psi_1(2)) = \psi_1^{(y-2)}(2\cdot 2) = \psi_1^{(y-2)}(2^2)$$
$$= \psi_1^{(y-3)}(\psi_1(2^2)) = \psi_1^{(y-3)}(2\cdot 2^2) = \psi_1^{(y-3)}(2^3)$$
$$= \dots = \psi_1^{(0)}(2^y) = 2^y$$

QUINDI

$$\boxed{\psi_2(y) = \psi_1^{(y)}(1) = 2^y = 2 \uparrow y}$$

SE $y > 0$,

$$\psi_2^{(y)}(1) = \psi_2^{(y-1)}(\psi_2(1)) = \psi_2^{(y-1)}(2)$$

$$= \psi_2^{(y-2)}(\psi_2(2)) = \psi_2^{(y-2)}(2\uparrow 2) = \psi_2^{(y-2)}(2\uparrow\uparrow 2)$$

$$= \psi_2^{(y-3)}(\psi_2(2\uparrow\uparrow 2)) = \psi_2^{(y-3)}(2\uparrow(2\uparrow\uparrow 2)) = \psi_2^{(y-3)}(2\uparrow\uparrow 3)$$

$$= \dots = \psi_2^{(0)}(2\uparrow\uparrow y) = 2\uparrow\uparrow y$$

QUINDI

$$\boxed{\psi_3(y) = \psi_2^{(y)}(1) = 2\uparrow\uparrow y = 2\uparrow^{(2)} y}$$

SE $y > 0$,

$$2\uparrow^{(3)}2 = 2\uparrow^{(2)}2$$

$$\begin{aligned}
\psi_3^{(y)}(1) &= \psi_3^{(y-1)}(\psi_3(1)) = \psi_3^{(y-1)}(2) \\
&= \psi_3^{(y-2)}(\psi_3(2)) = \psi_3^{(y-2)}(2\uparrow^{(2)}2) = \psi_3^{(y-2)}(2\uparrow^{(3)}2) \\
&= \psi_3^{(y-3)}(\psi_3(2\uparrow^{(3)}2)) = \psi^{(y-3)}(2\uparrow^{(2)}(2\uparrow^{(3)}2)) \\
&= \psi^{(y-3)}(2\uparrow^{(3)}3) \\
&= \dots = \psi_3^{(0)}(2\uparrow^{(3)}y) = 2\uparrow^{(3)}y
\end{aligned}$$

QUINDI

$$\boxed{\psi_4(y) = \psi_3^{(y)}(1) = 2\uparrow^{(3)}y}$$

PIÙ IN GENERALE SI HA:

$$\psi_{x+1}(y) = 2\uparrow^{(x)} y$$

$x \backslash y$	0	1	2	3	4	5	6	7	8	
0	1	2	4	5	6	7	8	9	10	...
1	1	$2\cdot 1$	$2\cdot 2$	$2\cdot 3$	$2\cdot 4$	$2\cdot 5$	$2\cdot 6$	$2\cdot 7$	$2\cdot 8$	...
2	1	$2\uparrow 1$	$2\uparrow 2$	$2\uparrow 3$	$2\uparrow 4$	$2\uparrow 5$	$2\uparrow 6$	$2\uparrow 7$	$2\uparrow 8$	...
3	1	$2\uparrow^{2} 1$	$2\uparrow^{2} 2$	$2\uparrow^{2} 3$	$2\uparrow^{2} 4$	$2\uparrow^{2} 5$	$2\uparrow^{2} 6$	$2\uparrow^{2} 7$	$2\uparrow^{2} 8$	...
4	1	$2\uparrow^{3} 1$	$2\uparrow^{3} 2$	$2\uparrow^{3} 3$	$2\uparrow^{3} 4$	$2\uparrow^{3} 5$	$2\uparrow^{3} 6$	$2\uparrow^{3} 7$	$2\uparrow^{3} 8$	...
5	1	$2\uparrow^{4} 1$	$2\uparrow^{4} 2$	$2\uparrow^{4} 3$	$2\uparrow^{4} 4$	$2\uparrow^{4} 5$	$2\uparrow^{4} 6$	$2\uparrow^{4} 7$	$2\uparrow^{4} 8$	...
6	1	$2\uparrow^{5} 1$	$2\uparrow^{5} 2$	$2\uparrow^{5} 3$	$2\uparrow^{5} 4$	$2\uparrow^{5} 5$	$2\uparrow^{5} 6$	$2\uparrow^{5} 7$	$2\uparrow^{5} 8$	...
7	1	$2\uparrow^{6} 1$	$2\uparrow^{6} 2$	$2\uparrow^{6} 3$	$2\uparrow^{6} 4$	$2\uparrow^{6} 5$	$2\uparrow^{6} 6$	$2\uparrow^{6} 7$	$2\uparrow^{6} 8$	...
8	1	$2\uparrow^{7} 1$	$2\uparrow^{7} 2$	$2\uparrow^{7} 3$	$2\uparrow^{7} 4$	$2\uparrow^{7} 5$	$2\uparrow^{7} 6$	$2\uparrow^{7} 7$	$2\uparrow^{7} 8$	...
...	...	...	...	...	...	...	...	...	...	...

$x \backslash y$	0	1	2	3	4	5	6	7	8	
0	1	2	4	5	6	7	8	9	10	...
1	1	$2\cdot 1$	$2\cdot 2$	$2\cdot 3$	$2\cdot 4$	$2\cdot 5$	$2\cdot 6$	$2\cdot 7$	$2\cdot 8$	...
2	1	$2\uparrow 1$	$2\uparrow 2$	$2\uparrow 3$	$2\uparrow 4$	$2\uparrow 5$	$2\uparrow 6$	$2\uparrow 7$	$2\uparrow 8$	...
3	1	$2\uparrow^{2} 1$	$2\uparrow^{2} 2$	$2\uparrow^{2} 3$	$2\uparrow^{2} 4$	$2\uparrow^{2} 5$	$2\uparrow^{2} 6$	$2\uparrow^{2} 7$	$2\uparrow^{2} 8$	...
4	1	$2\uparrow^{3} 1$	$2\uparrow^{3} 2$	$2\uparrow^{3} 3$	$2\uparrow^{3} 4$	$2\uparrow^{3} 5$	$2\uparrow^{3} 6$	$2\uparrow^{3} 7$	$2\uparrow^{3} 8$	...
5	1	$2\uparrow^{4} 1$	$2\uparrow^{4} 2$	$2\uparrow^{4} 3$	$2\uparrow^{4} 4$	$2\uparrow^{4} 5$	$2\uparrow^{4} 6$	$2\uparrow^{4} 7$	$2\uparrow^{4} 8$	...
6	1	$2\uparrow^{5} 1$	$2\uparrow^{5} 2$	$2\uparrow^{5} 3$	$2\uparrow^{5} 4$	$2\uparrow^{5} 5$	$2\uparrow^{5} 6$	$2\uparrow^{5} 7$	$2\uparrow^{5} 8$	...
7	1	$2\uparrow^{6} 1$	$2\uparrow^{6} 2$	$2\uparrow^{6} 3$	$2\uparrow^{6} 4$	$2\uparrow^{6} 5$	$2\uparrow^{6} 6$	$2\uparrow^{6} 7$	$2\uparrow^{6} 8$	...
8	1	$2\uparrow^{7} 1$	$2\uparrow^{7} 2$	$2\uparrow^{7} 3$	$2\uparrow^{7} 4$	$2\uparrow^{7} 5$	$2\uparrow^{7} 6$	$2\uparrow^{7} 7$	$2\uparrow^{7} 8$	...
...	...	...	...	...	...	...	...	...	...	...

PONIAMO: $$A(n) =_{df} \psi_n(n) = 2\uparrow^{(n-1)} n$$

INVERSA DELLA FUNZIONE DI ACKERMANN

$$\boxed{\alpha(n) =_{def} \min \{ m : 2 \uparrow^{(m-1)} m \geq n \}}$$

n	0	1	2	3	4	5	$\dots$	16	17	$\dots$	$\psi_4(4)$	$\psi_4(4)+1$
$\alpha(n)$	0	0	1	2	2	3	$\dots$	3	4	$\dots$	4	5

$\psi_0(0) = 1$

$\psi_1(1) = 2$

$\psi_2(2) = 2^2 = 4$

$\psi_3(3) = 2\uparrow^{(2)} 3 = 2 \uparrow 2 \uparrow 2 = 2^{2^2} = 2^4 = 16$

$$\begin{aligned}
\psi_4(4) = 2 \uparrow^{(3)} 4 &= 2 \uparrow^{(2)} 2 \uparrow^{(2)} 2 \uparrow^{(2)} 2 \\
&= 2 \uparrow^{(2)} 2 \uparrow^{(2)} 2^2 = 2 \uparrow^{(2)} 2 \uparrow^{(2)} 4 \\
&= 2 \uparrow^{(2)} 2^{2^{2^2}} = 2 \uparrow^{(2)} 2^{16} \\
&= 2 \uparrow^{(2)} 65536 = \underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}_{65536}
\end{aligned}$$

$$n < \psi_4(4) = 2^{\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{65536}} \longrightarrow \alpha(n) \leq 4 \quad !!$$

PUR VALENDO $\lim_{n \to \infty} \alpha(n) = +\infty$, PER TUTTI GLI SCOPI PRATICI $\alpha(n)$ PUÒ ESSERE APPROSSIMATA CON LA COSTANTE 4 !!